Lecture 22

May 6^{th}, 2004

Define $u^+ := \max\{u, 0\}$, $u^- := \min\{u, 0\}$. For a generalized function $u \in W^{1,2}(\Omega)$ we say $u \leq 0$ *on* $\partial\Omega$ if $u^+ \in W_0^{1,2}(\Omega)$. Similarly we say $u \leq v$ *on* $\partial\Omega$ if $u - v \leq 0$ on $\partial\Omega$. Finally define $\sup\limits_{\partial\Omega} u := \inf\{c : u \leq c \text{ on } \partial\Omega\}$.

Weak $\mathbf{L^2}$ Maximum Principle

We consider the divergence form equation

$$Lu := \mathrm{D}_i(a^{ij}\mathrm{D}_j u) + b^i \mathrm{D}_i u + cu = f,$$

with $c \leq 0$.

Theorem. *Suppose $u \in W^{1,2}(\Omega)$. Assume*

- $c \leq 0$
- *L strictly elliptic with $(a^{ij}) > \gamma \cdot I$, $\gamma > 0$*
- $||b^i||_{C^0(\Omega)} \leq \Lambda$
- $f \in W^{k,2}(\Omega)$

$$\textit{Then} \begin{cases} \textit{If } Lu \geq 0 \ \textit{ then } \sup_\Omega u \leq \sup_{\partial\Omega} u^+. \\ \textit{If } Lu \leq 0 \ \textit{ then } \inf_\Omega u \geq \inf_{\partial\Omega} u^-. \\ \textit{If } c = 0 \textit{ then the above holds with } |u| \textit{ instead of } u. \end{cases}$$

The last conclusion follows from the first two since in that case u and $-u$ each satisfy one inequality.

Proof. From the statement we have that u satisfies an inequality in the weak sense, the integral inequality

$$\forall \ v \in W_0^{1,2}(\Omega) \qquad -\int_\Omega a^{ij}\mathrm{D}_j u \mathrm{D}_i v + \int_\Omega (b^i \mathrm{D}_i u + cu)v \geq 0$$

$$\text{or} \qquad \int_\Omega a^{ij}\mathrm{D}_j u \mathrm{D}_i v \leq \int_\Omega b^i \mathrm{D}_i uv + \int_\Omega cuv.$$

Now restrict to v such that $u \cdot v \geq 0$. Since $c \leq 0$

$$\int_\Omega a^{ij}\mathrm{D}_j u \mathrm{D}_i v \leq \int_\Omega b^i \mathrm{D}_i uv \leq \Lambda \int_\Omega v|\mathrm{D}u|.$$

If $\sup_\Omega u > \sup_{\partial\Omega} u^+$ then choose $k \in \mathbb{R}$ such that $\sup_{\partial\Omega} u^+ \leq k < \sup_\Omega u$. Now pick a specific v, $v := (u-k)^+$. This v is 0 everywhere except where u exceed k, and in particular where it exceeds the supremum of the boundary values. Indeed we have $v \in W_0^{1,2}(\Omega)$ as well as

$$\mathrm{D}v = \begin{cases} \mathrm{D}u & \text{for } u > k \ \ (\text{there } v > 0) \\ 0 & \text{for } u \leq k \ \ (\text{there } v = 0) \end{cases}.$$

And so

$$\int_\Omega a^{ij}\mathrm{D}_j v \mathrm{D}_i v \leq \Lambda \int_\Gamma v|\mathrm{D}v|,$$

where $\Gamma := \mathrm{supp}\mathrm{D}v \subseteq \mathrm{supp}v$. Now by strict ellipticity the LHS majorizes $\lambda \int_\Omega |\mathrm{D}v|^2$ hence

$$\lambda ||\mathrm{D}v||^2_{L^2(\Omega)} = \lambda \int_\Omega |\mathrm{D}v|^2 \leq \Lambda \int_\Gamma v|\mathrm{D}v| \leq \Lambda ||v||_{L^2(\Gamma)} ||\mathrm{D}v||_{L^2(\Omega)}$$

by the Hölder Inequality (HI) (for $p = q = 2$) and therefore

$$||\mathrm{D}v||_{L^2(\Omega)} \leq c(\lambda, \Lambda) \cdot ||v||_{L^2(\Gamma)} = c \cdot \Big(\int_\Gamma v^2\Big)^{\frac{1}{2}} \leq c \cdot \Big(\Big\{\int_\Gamma (v^2)^{\frac{n}{n-2}}\Big\}^{\frac{n-2}{n}} \Big\{\int_\Gamma 1^{\frac{n}{2}}\Big\}^{\frac{2}{n}}\Big)^{\frac{1}{2}}$$

$$= c \cdot \mathrm{Vol}(\Gamma)^{\frac{1}{n}} ||v||_{L^{\frac{2n}{n-2}}(\Gamma)}$$

once again by the HI for $p = \frac{n}{n-2}$, $q = \frac{n}{2}$. On the other hand by the Sobolev Embedding $||v||_{L^{\frac{2n}{n-2}}(\Omega)} \leq C||\mathrm{D}v||_{L^2(\Omega)}$ and so over all

$$||v||_{L^{\frac{2n}{n-2}}(\Omega)} \leq C||\mathrm{D}v||_{L^2(\Omega)} \leq C||v||_{L^2(\Omega)} c \cdot \mathrm{Vol}(\Gamma)^{\frac{1}{n}} ||v||_{L^{\frac{2n}{n-2}}(\Omega)}$$

and therefore $\mathrm{Vol}(\Gamma)^{\frac{1}{n}} \geq \tilde{C}$ where the constant is independent of k ! (note $v \in L^2(\Omega)$). Let therefore $k \to \sup_\Omega u$. Then we see u must still attain its maximum on a set of positive measure! But then $\mathrm{D}v = \mathrm{D}u = 0$ there! Which in turn contradicts this previous bound on the volume of $\Gamma = \mathrm{supp}(\mathrm{D}v)$. So we conclude that there exists no $k \in [\sup_{\partial\Omega} u^+, \sup_\Omega u)$, in other words $\sup_{\partial\Omega} u^+ \geq \sup_\Omega u$. The second case of the Theorem follows now since if $Lu \leq 0$ then $L(-u) \geq 0$ and the first case applies. ■

Corollary. *Let L be strictly elliptic with $c \leq 0$. Assume $u \in W_0^{1,2}(\Omega)$ satisfies $Lu = 0$ on Ω. Then $u = 0$ on Ω.*

An a priori Estimate

We improve slightly on the aesthetics of the higher regularity proved in the previous lecture for the case $c \leq 0$.

Theorem. *Let $u \in W_0^{1,2}(\Omega) \cap W^{k+2,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, and assume*

- *L strictly elliptic with $(a^{ij}) > \gamma \cdot I$, $\gamma > 0$*
- *$a^{ij} \in \mathcal{C}^{k,1}(\bar{\Omega})$*
- *$b^i, c \in \mathcal{C}^{k-1,1}(\bar{\Omega})$ (for $k = 0$, $\mathcal{C}^{-1,1} := \mathcal{C}^0 = L^\infty$)*
- *$f \in W^{k,2}(\Omega)$*
- *$\partial\Omega$ is $\mathcal{C}^{k+2}$*

Then

$$||u||_{W^{k+2,2}(\Omega)} \leq c \cdot ||Lu||_{W^{k,2}(\Omega)}.$$

Note that the assumption $u \in W^{k+2,2}(\Omega)$ is *superfluous* once $u \in W_0^{1,2}(\Omega)$ in light of our previous results.

Also note that this is exactly analogous to what we did in our Hölder theory study; there we proved $Lu = f \in \mathcal{C}^{k,\alpha}(\Omega),\ c \leq 0$ implies $||u||_{C^{k+2,\alpha}(\Omega)} \leq c||f||_{C^{k,\alpha}(\Omega)}$.

Proof. **Case** $k = 0$**.** We want to prove $||u||_{W^{2,2}(\Omega)} \leq c \cdot ||Lu||_{W^{2,2}(\Omega)}$. and we already know that

$$||u||_{W^{2,2}(\Omega)} \leq c \cdot \big(||u||_{L^2(\Omega)} + ||Lu||_{W^{2,2}(\Omega)}\big),$$

so we now try to demonstrate $||u||_{L^2(\Omega)} \leq c||Lu||_{W^{2,2}(\Omega)}$ for all $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. If not, pick a sequence $\{u_m\} \subseteq W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ with $||u_m||_{L^2(\Omega)} = 1,\ \ ||Lu_m||_{W^{2,2}(\Omega)} \xrightarrow{m\to\infty} 0$ and hence by what we know

$$||u_m||_{W^{2,2}(\Omega)} \leq c.$$

Since $W^{2,2}(\Omega)$ is a Hilbert space exists a subsequence which converges weakly to $u \in W^{2,2}(\Omega)$ (note Alouglou's Theorem applies as we have separability and every Hilbert space is a reflexive Banach space). Since $W^{2,2}(\Omega) \hookrightarrow L^2(\Omega)$ is a compact embedding we actually have $u_m \to u \in L^2(\Omega)$ (i.e strongly). But now $||Lu_m||_{L^2(\Omega)} \to 0$, hence $Lu = 0$ weakly. Since $c \leq 0$ this implies by our previous work $u = 0$! In contradiction with $||u_m||_{L^2(\Omega)} = 1$ as $u_m \to u$ in $L^2(\Omega)$ so $||u||_{L^2(\Omega)} = 1$ allora ... ■

Corollary. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with $\mathcal{C}^{k+2}$ boundary. Then the map*

$$\Delta\ : W^{k+2,2}(\Omega) \cap W_0^{1,2}(\Omega) \longrightarrow W^{k,2}(\Omega)$$

is an isomorphism.

Proof. **Injective:** By the previous Corollary if $L(u_1 - u_2) = 0$ on Ω and $u_1 - u_2 \in W_0^{1,2}(\Omega)$ then $u_1 - u_2 = 0$. This actually applies also to any two such functions in $W^{1,2}(\Omega)$ with equal boundary values.

Surjective: Let $f \in W^{k,2}(\Omega)$. We can find a solution $Lu = f$ with u in $W_0^{2,2}(\Omega)$ by Riesz Representation Theorem and our regularity theory. So Δ^{-1} exists and by our above Theorem satisfies

$$||\Delta^{-1} f||_{W^{k+2,2}(\Omega)} \leq C \cdot ||f||_{W^{k,2}(\Omega)}.$$

So Δ^{-1} is continuous. From the definition of Δ we see that

$$||\Delta u||_{W^{k,2}(\Omega)} \leq ||u||_{W^{k+2,2}(\Omega)}$$

(note no constant on RHS) we see also Δ itself is a continuous map between those spaces (WRT to their topologies). ■

Corollary. *For appropriate L (see above Theorems) with $c \leq 0$*

$$L : \ W^{k+2,2}(\Omega) \cap W_0^{1,2}(\Omega) \longrightarrow W^{k,2}(\Omega)$$

is an isomorphism.

Proof. **Injective:** Exactly as above.

Surjective: We employ the Continuity Method (CM) which will work out exactly as in the Schauder case. Consider the family of equations

$$L_t u := (1-t)\mathrm{D}u + tLu = f.$$

Recall that the CM will provide for the surjectivity of L based on the surjectivity of Δ (proved above) once we can prove

$$||u||_{W^{k+2,2}(\Omega)} \leq c \cdot ||L_t u||_{W^{k,2}(\Omega)}$$

with c independent of t. And this is indeed the case since each of the L_t satisfies the assumptions of the previous Theorem. ■

Negative Sobolev Spaces

What happens for the $k = -1$ case? Where does Δ map to? Δu is not defined as a function, though it is as a distribution: given $v \in W_0^{1,2}(\Omega)$ one can define

$$\Delta u(v) := -\int_\Omega \nabla u \cdot \nabla v$$

which realizes Δu as a linear functional on $W_0^{1,2}(\Omega)$, in other words

$$\Delta : \ W_0^{1,2}(\Omega) \longrightarrow (W_0^{1,2}(\Omega))^\star.$$

The motivation for this definition lies in the fact that when we look at the equation $-\int_\Omega \nabla u \cdot \nabla v = \int_\Omega \Delta u v$ we actually mean $\int_\Omega v \cdot (\Delta u d\mathbf{x})$ and $\Delta u d\mathbf{x}$ gives a distribution under the identification of distributions with measures.

Recall the inner product as we defined it in $W_0^{1,2}(\Omega)$ is

$$(u, v) = +\int_\Omega \nabla u \cdot \nabla v.$$

By the Riesz Representation Theorem given any element $F \in (W_0^{1,2}(\Omega))^\star$ there exists a unique $u \in W_0^{1,2}(\Omega)$ such that $F(v) = (u, v)$, so

$$F(v) = (u, v) = +\int_\Omega \nabla u \cdot \nabla v = (-\Delta u)(v),$$

as distributions. Therefore Δ is surjective. Injectivity follows from the definition of Δ. Continuity of the inverse is also provided for by the Riesz Representation Theorem

$$||u||_{W_0^{1,2}(\Omega)} = ||-\Delta u||_{(W_0^{1,2}(\Omega))^\star}.$$

We conclude from this short discussion that $\Delta : W_0^{1,2}(\Omega) \longrightarrow (W_0^{1,2}(\Omega))^\star =: W^{-1,2}(\Omega)$ is an isomorphism of Hilbert Spaces. This is a natural extension to our previous results, and adopting this notation they all extend now to the case $k = -1$.